Finite temperature transport at the superconductor-insulator transition in disordered systems

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Abstract: I argue that the incoherent, zero-frequency limit of the universal crossover function in the temperature-dependent conductivity at the superconductor-insulator transition in disordered systems may be understood as an analytic function of dimensionality of the system d, with a simple pole at $d = 1$. Combining the exact result for the crossover function in $d = 1$ with the recursion relations in $d = 1 + \epsilon$, the leading term in the Laurent series in the small parameter ϵ for this quantity is computed for the systems of disordered bosons with short-range and with Coulomb interactions. The universal, low temperature, dc critical conductivity for the dirty boson system with Coulomb interactions in $d = 2$ is estimated to be $\sim 0.69(2e)^2/h$, in good agreement with many experiments on thin films. The next order correction is likely to somewhat increase the result, possibly bringing it closer to the self-dual value.

Different quasi two-dimensional (2D) electronic systems, like thin films [1], Josephson junction arrays [2] or underdoped high-Tc cuprates [3], appear to have a continuous zero-temperature phase transition between the superconducting and the insulating state, as some parameter of the system is varied. The loss of phase coherence in the ground state is believed to be due to Anderson localization of the Cooper pairs [4], [5], which through the quantum uncertainty relation competes with the phase ordering. In reality, this purely quantum ($T = 0$) phenomenon is unavoidably observed at low but finite temperatures, typically as a crossover from increasing to decreasing dc conductivity with temperature. Particularly interesting is the behavior of the conductivity near the critical value of the tuning parameter, which remains finite down to the lowest temperatures, and usually very close to $(2e)^2/h$, the quantum unit of conductance for electron pairs. This near-universality of the metallic transport right at the superconductor-insulator (SI) transitions in 2D has been a subject of numerous theoretical and experimental investigations within the last decade. Scaling arguments [6] imply that at $T = 0$ and right at the quantum critical point the dc conductivity must indeed be universal, very much like the critical exponents. This insight has received a substantial amount of theoretical support in form of concrete numerical [7], [8], [9] and analytical [7], [10], [11] evaluations of the $T = 0$ critical conductivity for various universality classes of the SI transitions. The experimental evidence for the expected universality of the charge transport however, is still somewhat less convincing. In particular, measurements typically yield values of the critical conductivity two to three times larger than the calculations [1]. Most of the theoretical studies [7], [8], [9], [10], [11] of the critical conductivity however, are in significant conceptual discord with the experiment, as recently emphasized by Damle and Sachdev [12]. They argued that the $\omega = 0$, $T \to 0$ dc conductivity that is typically measured and the $T = 0$, $\omega \to 0$ conductivity that is usually calculated, originate in entirely different dissipation mechanisms, and while both should be universal, they have no reason to be equal. As an illustration, they obtained the conductivity in the hydrodynamic, incoherent transport regime $\hbar\omega/k_B T \to 0$ at the simpler superfluid-Mott insulator (SF-MI) transition brought by an external periodic potential, near its upper critical dimension of $d = 3$, and indeed found a different, and larger, value of the critical conductivity.

In real disorder systems the insulating phase is presumably the result of Anderson localization and should therefore be a compressible Bose-glass (BG) [5], not the Mott insulator. It has been argued that, with disorder present, the limits $\omega \to 0$ and $T \to 0$ may commute [7], and the imaginary time Monte Carlo calculations indeed found little or no dependence of the critical conductivity at the SF-BG transition on the ratio ω_n/T [9], where ω_n is the Matsubara frequency. Including disorder makes the problem difficult to study analytically, since then the transition seems to lack the upper critical dimension, or any other obvious limit in which at the criticality the system would be weakly coupled. Numerical techniques [7], [8], [9] on the other hand, determine the conductivity by extrapolating from the imaginary Matsubara frequencies, $\omega_n = n2\pi k_B T$, and thus by its nature are not very sensitive to any structure it might have in the hydrodynamic regime. The understanding of the role of disorder and the calculation of the experimentally measured low temperature dc conductivity at the SI critical point in 2D systems thus presents itself as a fundamental and unsolved prob-

lem. The purpose of this Letter is to propose its solution in form of a controlled expansion around the solvable case of the SF-BG transition in one dimension (1D).

I will consider the system of *disordered* interacting bosons of charge $e_* = 2e$ defined, for example, by the Bose-Hubbard, or Josephson junction array Hamiltonian with a random chemical potential [5], [11]. This model is known to possess a continuous SF-BG transition at $T = 0$, and should be appropriate for description of the SI transition observed in Josephson junction arrays, or ^4He in random media. If the Coulomb interaction between bosons is added, it should represent the correct universality class for the transition in homogeneous and granular films. Although there is no upper critical dimension for the SF-BG transition in the usual sense [13], [5], [14] the fact that the superfluid phase in $d = 1$ and at $T = 0$ exhibits only a power-law long-range order implies that $d = 1$ represents the lower critical dimension. Recently, this observation has been used by the author to formulate a controlled expansion of the universal quantities at the SF-BG transition in powers of a small parameter $\epsilon = d - 1$ [14]. In particular, and as argued below, in the limit $T \to 0$ the dc resistivity at the critical point is proportional to a certain power of temperature and to the value of the disorder parameter at the SF-BG critical point, which becomes infinitesimally small ($\sim \epsilon$) as dimensionality of the system is reduced to $d = 1$. This suggests that the universal part of the low temperature dc conductivity at the critical point may be expressed as an analytic function of ϵ, with a simple pole at $\epsilon = 0$. I compute the first term in the Laurent series for the real part of the critical dc conductivity around $\epsilon = 0$:

$$\sigma_c'(\omega = 0, T) = (\frac{\hbar c}{k_B T})^{\frac{2-d}{z}} [\frac{6x}{\pi^{5/2}} \frac{1}{\epsilon} + O(1)] \frac{e_*^2}{h}, \tag{1}$$

where c is a microscopic constant with units $(length)^z/time$, $z = \{d, 1\}$ is the dynamical exponent and $x = \{1, 2\}$ for the short-range and the Coulomb interaction between bosons, respectively. For $d = 2$ this leads to an estimate $\sigma_c \approx 0.69 e_*^2/h$ for the Coulomb universality class, in agreement with many experiments on thin films [1]. A systematic perturbative procedure for a further calculation of the critical conductivity and of the critical exponents is outlined, and the sign of the next order correction in the Eq. (1) is discussed. I speculate that at the SF-BG critical point in 2D the real part of the conductivity is a continuously decreasing function of the ratio $\hbar\omega/k_B T$, qualitatively similar to the result in $d = 1 + \epsilon$.

In general, right at the critical point in a d-dimensional system the real part of conductivity for the frequencies and the temperatures much smaller then some microscopic cutoff energy can be written as

$$\sigma_c'(\omega, T) = (\frac{\hbar c}{k_B T})^{\frac{2-d}{z}} F_d(\frac{\hbar\omega}{k_B T}) \frac{e_*^2}{h}, \tag{2}$$

where $F_d(x)$ is a *universal* crossover function. While I focus here on the SI transition, the above equation applies equally well at the critical points in quantum Hall systems [15] and doped semiconductors [16], for example. At $d = 2$ the power of the temperature-dependent dephasing length [15] in the last equation vanishes, and the conductivity in units of e_*^2/h becomes a universal function of the ratio of quantum and thermal energy scales.

It then becomes evident that, in principle, at the critical point the conductivity in $d=2$ assumes two completely different values depending on the order of limits $\omega \to 0$ and $T \to 0$, that correspond to the values of the crossover function either at zero or at infinity, and measure completely incoherent or completely coherent transport, respectively. What is more surprising, although it arises as a natural consequence of the scaling law, is that even in the limit $\hbar\omega/k_B T \to 0$ of incoherent, collision dominated transport, conductivity at the critical point in $d=2$ is still given by a *universal* number [12]. To understand the dependence of the conductivity in this limit on dimensionality of the system, consider the effective low-energy action for the disordered superfluid that describes the SF-BG transition in the dirty Bose-Hubbard model in $d=1$ [17], [11], [14]:

$$S = \frac{K}{\pi}\sum_{i=1}^{N}\int dx \int_0^\beta d\tau [c^2(\partial_x\theta_i(x,\tau))^2 + (\partial_\tau\theta_i(x,\tau))^2]$$
$$-D\sum_{i,j=1}^{N}\int dx \int_0^\beta d\tau d\tau' \cos 2(\theta_i(x,\tau) - \theta_j(x,\tau')), \tag{3}$$

where $\beta = \hbar/k_B T$, K is inversely proportional to the superfluid density, c is the velocity of low-energy phononic excitations, index i numerates the replicas introduced to average over disorder, and D is related to the width of the distribution of the random potential. The usual limit $N \to 0$ and a short-distance cutoff Λ^{-1} in the Eq. (3) are assumed. This effective action arises as the one dimensional realization of the density (dual) representation of the dirty Bose-Hubbard model at low energies [11], or as the bosonic representation of the disordered Luttinger liquid [17]. Invariance under a change of the cutoff implies that the conductivity in $d=1$ can be written in the scaling form

$$\sigma(\omega, T) = \frac{\hbar c}{k_B T} f(K(b), c(b), D(b), \frac{\hbar\omega}{k_B T})\frac{e_*^2}{h}, \tag{4}$$

where $K(b)$, $c(b)$ and $D(b)$ are the renormalized couplings at the new cutoff $b\Lambda^{-1}$, and $b = \hbar c\Lambda/k_B T$. At low temperatures the conductivity is determined by the infrared stable fixed point of the scaling transformation. The result of the renormalization in the theory (3) for weak disorder is well known [17], [14]: under the change of cutoff the combination $\kappa = Kc^2$, that is proportional to the compressibility, stays constant, K always increases, and small D is relevant for $\eta = Kc > 1/3$, and irrelevant otherwise. There exists a separatrix in the $\eta - D$ plane which ends in the SF-BG critical point at $\eta = 1/3$ and $D = 0$. At $\omega = 0$ the scaling function in the Eq. (4) at weak disorder should behave as $f \sim 1/D(b)$ [17], [18], and therefore right at the separatrix slowly (logarithmically, $\sim (\ln(b))^2$, [18]) diverges as $b \to \infty$ in $d=1$. Apart from the logarithmic correction that derives from disorder being dangerously irrelevant at the SF-BG criticality in 1D, right at the transition the Eq. (4) is just a special case of the general scaling form (2), for $d = z = 1$. Since in $d = 1+\epsilon$ disorder at the transition scales towards a finite fixed point value $D(b) \to D^* \sim \epsilon$, when $b \to \infty$ [14], the comparison of the two scaling laws in Eqs. (2) and (4) leads to the identification

$$F_{1+\epsilon}(0) = f(1/3, 1, D^*, 0) = \frac{const.}{\epsilon} + O(1), \tag{5}$$

which expresses the central idea of this work. The leading term in the Laurent series for $F_d(0)$ is completely determined by the scaling function as in 1D and by the infinitesimal value of disorder at the fixed point of the scaling transformation in $d = 1 + \epsilon$.

In the remaining of the paper the complete function f near the SF-BG critical point in $d = 1$ is obtained, a new field-theoretic version of the recursion relations in $d = 1 + \epsilon$ requisite for determination of the fixed point value of disorder is derived, and finally, the residuum at the pole at $\epsilon = 0$ in the Eq. (5) for both short-range and Coulomb interactions between bosons is computed.

The standard linear response formalism yields the conductivity in $d = 1$

$$\sigma(\omega, T) = -i\frac{2\omega}{\pi}\frac{e_*^2}{h}\lim_{N\to 0}\frac{1}{N}\sum_{n,m=1}^{N} G^r_{nm}(\omega). \tag{6}$$

$G^r_{nm}(\omega)$ is the temperature dependent, retarded, $q = 0$ Green's function defined as

$$G^r_{nm}(\omega) = \int dx \int_0^\beta d\tau e^{i\omega_n\tau}\langle T_\tau \theta_n(x,\tau)\theta_m(0,0)\rangle_{i\omega_n\to\omega+i\delta}, \tag{7}$$

where T_τ is the standard time-ordering operator. The thermal Green's function in the Eq. (7) may be evaluated perturbatively in disorder and then analytically continued to real frequencies. Similar calculation has been performed before by Luther and Peschel [19] for $\eta > 1$, which corresponds to weak coupling in the equivalent 1D fermionic system. The SF-BG transition at weak disorder is at $\eta \approx 1/3$, so I derive here a slightly improved version of their results which can be analytically continued into the transition region. Introduce the self-energy as $G^t_{ij}(\omega_n) = \delta_{ij}/(2(K/\pi)\omega_n^2 + \Sigma^t(\omega_n))$. To the lowest order in D it may be written as

$$\Sigma^t(\omega_n) = 8D\int_0^\beta d\tau(1 - e^{i\omega_n\tau})\langle T_\tau e^{i2\theta_j(x,\tau)}e^{-i2\theta_j(x,0)}\rangle_0 + O(D^2), \tag{8}$$

where the average is performed over the quadratic part of the action (3). The self-energy is thus *itself* a Green's function, which enables one to perform the analytic continuation to real frequencies by first rotating the integrand in (8) to real time by $\tau \to it$ to find the real time time-ordered propagator, and from it finally to determine the retarded one by using the standard relation between them [20]. Performing the Fourier transform at the resulting expression then gives the retarded self-energy

$$\Sigma^r(\omega) = \frac{16\pi D}{c\Lambda}\left(\frac{\pi k_B T}{\hbar c\Lambda}\right)^{\frac{1}{\eta}-1}\sin\left(\frac{\pi}{2\eta}\right)e^{\frac{C}{\eta}}\int_0^\infty dt\frac{1 - e^{i(\frac{\hbar\omega}{k_B T})t}}{(\sinh(\pi t))^{\frac{1}{\eta}}}, \tag{9}$$

where $C \approx 0.577$ is the Euler's constant, and I assumed that $\hbar c\Lambda/k_B T >> 1$, i.e. the continuum (low-temperature) limit. Appearance of a particular numerical constant is a consequence of the assumption that the dispersion is $\omega = ck$ for all momenta $0 < k < \Lambda$, and is a non-universal, short-distance feature. This nevertheless, does not compromise the universality of the conductivity at the transition, since any non-universal constant like C may at the end be absorbed into the definition of the running, dimensionless disorder coupling,

as will be done shortly. The remaining integral in (9) is convergent only for $\eta > 1$, but once evaluated there exactly, may be defined via analytic continuation in the transition region $\eta \approx 1/3$. Performing the integral, in the vicinity of $\eta = 1/3$ one obtains the conductivity in $d = 1$ to be

$$\sigma(\omega, T) = \frac{i\omega c}{\eta(T)\omega^2 + i2(k_B T/\hbar)^2 W(T) g(\hbar\omega/k_B T)} \frac{e_*^2}{h}, \tag{10}$$

where $W(T) = (\pi^4 D/c^2\Lambda^3 e^{3C})(k_B T/\hbar c\Lambda)^{\frac{1}{\eta}-3}$ is the dimensionless disorder variable, $\eta(T) = \eta + W(T)\pi^{-3/2}\tan(\pi/2\eta)$, and $g(x) = (1 + (x/\pi)^2)\tanh(x/2)$. Note that the result indeed may be cast into the scaling form as claimed in the Eq. (4). The self-energy acquired an imaginary part, which in the dc limit $\hbar\omega/k_B T \to 0$ becomes proportional to temperature and to the temperature dependent disorder variable $W(T)$. The significance of the point $\eta = 1/3$ now becomes apparent: for $\eta < 1/3$ disorder variable $W(T)$ scales towards zero with decreasing temperature, and the lowest order result in the Eq. (10) becomes asymptotically exact. At a finite frequency and at $T = 0$ the real part of conductivity in $d = 1$ then becomes $\sigma'(\omega) = (\pi c/\eta(0))\delta(\omega)$ and the system is an ideal conductor. If $\eta > 1/3$ the perturbation theory breaks down, which indicates the entrance into the insulating phase. Notice that as $\eta \to 1/3^-$ the coefficient in front of $W(T)$ in the expansion for $\eta(T)$ becomes divergent, as it has a simple pole at $\eta = 1/3$. This is reminiscent of the dimensional regularization frequently employed in the studies of thermal critical phenomena [21], and indicates that the theory defined by the action in Eq. (3) becomes just renormalizable at $\eta = 1/3$.

In the continuum limit, the effect of change of temperature on the low-frequency conductivity in $d = 1$ may be expressed entirely through the effective values of the coupling constants $\eta(T)$, $W(T)$, and, if we had retained the momentum dependence of the propagator, the compressibility $\kappa(T)$. This, of course, is just the statement of renormalizability of the theory, with the temperature used as an infrared regulator. Taking into account the non-zero canonical dimensions of the coupling constants away from $d = 1$ [14], the effective couplings close to $\eta = 1/3$ satisfy the differential equations:

$$\dot{\eta}(T) = z^{-1}(d-1)\eta(T) - \frac{2}{\pi^{5/2}} W(T) + O(W^2(T)), \tag{11}$$

$$\dot{W}(T) = (\frac{1}{\eta} - 3)W(T) + O(W^2(T)), \tag{12}$$

$$\dot{\kappa}(T) = z^{-1}(d-z)\kappa(T), \tag{13}$$

where $\dot{x} = dx/d\ln(k_B T/\hbar c\Lambda)$. The d-dependent terms in Eqs. (11) and (13) may be inferred from the scaling of the superfluid density and the compressibility near the critical point [5], as discussed at length elsewhere [14]. Note that in the Eq. (13), unlike in the Eq. (11), there are no terms proportional to $W(T)$. This is a consequence of the exact symmetry of the interaction term in the action (2) under $\theta_i(x,\tau) \to \theta_i(x,\tau) + h(x)$, for arbitrary function $h(x)$, and fixes the value of dynamical exponent to $z = d$ for the system with short-range interactions [14]. Linearization of the flow close to the fixed point of Eqs. (11) and (12) gives the correlation length exponent $\nu = (1/\sqrt{3\epsilon}) + O(1)$, in agreement with

the result of the momentum-shell renormalization group [14]. The fixed point is located at $W^*(T) = \pi^{5/2}\epsilon/6 + O(\epsilon^2)$ and $\eta^*(T) = 1/3 + O(\epsilon)$. Using the Eqs. (10) and (5) one then obtains the main result announced in the Eq. (1), for the short-range interactions between bosons.

To make a comparison with experiments on thin films [1] or on high-Tc cuprates [3] one needs to include the long-range Coulomb repulsion between the electron pairs. A way to do this was proposed previously by the author in the ref. 14, where the long-range interactions was defined as $V(\vec{r}) = e^2 \int d^d\vec{q} \exp(i\vec{q}\cdot\vec{r})/q^{d-1}$, so that it coincides with the Coulomb interaction for $d > 1$, and with the short-range interaction precisely at $d = 1$. The calculation of conductivity in $d = 1$ then remains the same, the only change now being the equation for the temperature dependent charge $e^2(T)$ instead for the compressibility in the Eq. (13), with $(z - d) \to (z - 1)$, and consequently, $(d - 1) \to (d - 1)/2$ in the Eq. (11) [14]. It then follows that with Coulomb interactions present $z = 1$, $\nu = \sqrt{2/3\epsilon} + O(1)$, and the fixed point value $W^*(T) = \pi^{5/2}\epsilon/12 + O(\epsilon^2)$. This yields the second result in the Eq. (1), for the Coulomb universality class. To the lowest order, the critical dc conductivity is larger if the Coulomb interaction is present. This may have been intuitively expected: a longer-range interaction suppresses the phase order more efficiently, so it takes less disorder to finally turn the system into an insulator.

Although there is no very good agreement on the value of critical conductivity between the different experiments, most of the measurements [1] on thin films are very close to $\sigma_c \approx 1(2e)^2/h$, in a quite reasonable agreement with my lowest order estimate for the Coulomb universality class. On physical grounds, one may also expect the next order term in the Eq. (11) to be again negative [14], and that there is no second order correction in D in the imaginary part of the self-energy [18]. The first expectation is based on the fact that disorder inhibits superfluidity, and therefore should effectively increase the exponent (η) in the power-law for the superfluid correlator at large distances. Also, in 1D the coupling D measures the $2k_F$ backscattering over the random potential in the equivalent fermionic problem, so D^2 corrections should renormalize only the forward scattering amplitude, which in 1D is unrelated to conductivity. In that case the next order $O(1)$ correction in the Eq. (1) would be positive, presumably bringing the result closer to the experimental one. It is also interesting to note that for both universality classes the critical conductivity in the hydrodynamic regime obtained here turns out to be larger than the one in the coherent, $T = 0$ limit [9], [11]. This also appears to be true for the SF-MI transition in a commensurate periodic potential [12]. Since $d = 1$ represents the lower critical dimension for the SF-MI transition as well, a similar calculation to the present one could presumably be performed in that case, except that it would require a calculation of the second order in strength of the periodic potential. It would be interesting to compare the result obtained this way with the calculation near the upper critical dimension of the ref. 12.

Even though the Eq. (10) has been derived here with the purpose of obtaining the universal conductivity in $d = 1 + \epsilon$, it is directly applicable to transport near the possible transition in two edges of the incompressible quantum liquid in a gated Hall bar [22]. Also, by continuity, the Eq. (10) implies that the crossover function $F_d(x)$ is a continuously

decreasing function of its argument for $d = 1 + \epsilon$, with a maximum $\sim 1/\epsilon$ at $x = 0$, and vanishing as $\sim \epsilon/x^{(1-\epsilon)/z}$ for large x. Although not ruled out, it does seem unlikely that in $d = 2$ this dependence on $x = \hbar\omega/k_B T$ should completely disappear. In fact, the difference in estimated critical conductivities in completely coherent and incoherent regimes suggests that the situation in $d = 2$ is most likely qualitatively similar to $d = 1 + \epsilon$: the real part of conductivity should continuously decrease as a function of $\hbar\omega/k_B T$, interpolating between the two finite, $\omega = 0$ and $T = 0$, limits.

Finally, the field-theoretic formulation of the renormalization group transformation derived here has the advantage over the usual momentum-shell calculation [17], [14] in that it facilitates a more systematic higher-order calculation. The observation that η in $d = 1$ plays a role similar to dimensionality in the classical critical phenomena suggests a procedure analogous to the standard dimensional regularization for the d-independent part of the recursion relations. Adding the effect of dimensionality when $d > 1$ as described in ref. 14 and as done in Eqs. (11) - (13) would then yield the higher order corrections for the exponent ν and the critical dc conductivity. It would be very interesting to compare the results of such an analytical calculation with the experiments and the numerical simulations, as it could lead to a more definite understanding of the SF-BG quantum critical behavior.

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